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Hamiltonian versus Lagrangian formulations of supermechanics

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Abstract. We take advantage of different generalizations of the tangent manifold to the context of graded manifolds, together with the notion of super section along a morphism of graded manifolds, to obtain intrinsic definitions of the main objects in supermechanics such as, the vertical endomorphism, the canonical and the Cartan's graded forms, the total time derivative operator and the super-Legendre transformation. In this way, we obtain a correspondence between the Lagrangian and the Hamiltonian formulations of supermechanics.

1. Introduction

The idea of considering classical systems that incorporate commuting and anticommuting variables to study dynamical systems dealing with bosonic and fermionic degrees of freedom, in particular supermechanics, has been in the air for some time now. Moreover, it has proved to be quite useful, not only in physics but also in mathematics. Nevertheless, a careful study of the geometric foundations of supermechanics was not taken very seriously, or at least people did not pay the necessary attention, until quite recently [12], in spite of the general tendency to geometrize physics. One of the reasons for this is that although the general consensus is that the proper setting is the theory of supermanifolds, there is no general agreement, for instance, as to what the velocity phase space of the system should be, since there are several different possibilities to generalize the concept of the tangent bundle in the context of graded manifolds. One of the central points in [12] was the introduction of the tangent supermanifold, which proved to be the right arena to develop Lagrangian supermechanics, since it allowed an intrinsic theory. However, some of the central objects, although well defined, were not defined in an intrinsic way. Perhaps the main drawback of the tangent supermanifold is that it is not a bundle. To overcome this, we enlarge this tangent supermanifold by considering the tangent superbundle as introduced by Sánchez-Valenzuela in [18], which unfortunately is a little too big, as its dimension is $(2m+n, 2n+m)$ if the dimension of the starting graded manifold (the superconfiguration space) is (m, n), but that has the big advantage of allowing a geometric interpretation of a supervector field as a section of a superbundle in much the same way as in non-graded geometry. We shall show in this paper the convenience of getting a compromise between both concepts: we shall introduce the objects using the tangent superbundle approach, but thereafter we shall read the results in the tangent supermanifold (identified as a subsupermanifold of the tangent superbundle). It will be shown how the tangent superbundle structure is the appropriate framework for an intrinsic definition of objects such as the total time derivative operator, the vertical superendomorphism, the Cartan 1-form and, fundamentally, the Legendre transformation, which will allow us to establish a connection between the Lagrangian and the Hamiltonian formalisms of supermechanics, similar to the one in classical mechanics.

In the geometrical approach to classical mechanics, the infinitesimal transformations arising in the traditional approach are described by the flow of vector fields, which can be considered either as sections of the tangent bundle, or as derivations of the commutative algebra $C^\infty(M)$ of differentiable functions. The generalization of the concept of a flow of a supervector field is not an easy task [16], but the corresponding idea of a vector field translates easily to the framework of supermechanics. It was shown in [6] that in order to incorporate non-point transformations in velocity phase space, it is necessary to introduce the concept of a supervector field along a map. Moreover, the use of such a concept and its generalizations, sections of a vector bundle along a map, has proved to be very useful for a better understanding of many aspects of classical mechanics [7, 8]. What we want to show is that in the transition to the supermechanics setting these concepts are even more necessary because of the inconvenience of working with points in graded geometry. Therefore, in the process of constructing a geometrical approach to supermechanics, including fermonic degrees of freedom, one of the first concepts to be introduced is that of a section along a morphism of supermanifolds.

The organization of the paper is as follows. In section 2 we describe the tangent superbundle, in particular we give a 'Batchelor's description' of it, and discuss its relation to the tangent supermanifold as defined by Ibort and Marín-Solano in [12]; it is shown that supervector fields can then be seen as geometric sections of the tangent superbundle. In section 3, we introduce the notion of a section along a morphism of graded manifolds, and represent supervector fields along a morphism as geometric sections along the morphism of the tangent superbundle. As a particular example, we give an intrinsic definition of the total time derivative operator that was used in [4] to obtain a version of Noether's theorem in supermechanics. This plays an important role in the geometry of the tangent superbundle, and thereby in the Lagrangian formalism of supermechanics.

Section 4 is devoted to the study of graded forms along a morphism of graded manifolds. Furthermore, we study the canonical graded 1-form, Θ_0, on the supercotangent manifolds, as well as the degeneracy of the graded form $\Omega_0 = -\mathrm{d}\Theta_0$. Finally, section 5 is concerned with the vertical superendomorphism, which is necessary to introduce the Cartan 1-form corresponding to a Lagrangian superfunction, and also with the super Legendre transformation. Finally using the machinery developed here, we establish a relationship between the Lagrangian and the Hamiltonian formulations of supermechanics.

2. The tangent superbundle and the tangent supermanifold

2.1. Basic notation

At the heart of the graded manifold theory is the idea of equipping a supervector space $V = V_0 \oplus V_1$ with the structure of graded manifold; the natural way of doing this [13, 14] is to consider the so-called affine supermanifold:

$$S(V) := \Big(V_0, C^\infty(V_0) \otimes \bigwedge(V_1^*)\Big). \tag{2.1}$$

Nevertheless, this has some drawbacks from the categorical point of view [19], and, in the context of supervector bundles, Sánchez-Valenzuela realized that, instead of the affine supermanifold, it is more appropriate to use the supermanifoldification of V [18, 3]:

$$V_S := S(V \oplus \Pi V) \tag{2.2}$$

where Π is the change of parity function [14, 15], hence $(\Pi V) =)\Pi V)_0 \oplus (\Pi V)_1$, where

$$(\Pi V)_i = V_{i+1} \qquad i = 0, 1. \tag{2.3}$$

The sheaf, $C^\infty(V_0) \otimes \bigwedge(V_1^*)$, will be denoted by $\mathcal{A}_{m,n}$ whenever $\dim V_0 = m$, $\dim V_1 = n$, and $\mathbb{R}^{m|n}$ will denote the graded manifold $\mathbb{R}^{m|n} = (\mathbb{R}^m, \mathcal{A}_{m,n})$. On the other hand, we shall always consider, on $\mathbb{R}^{m+n|m+n} = (\mathbb{R}^m \oplus \mathbb{R}^n)_S$, the following supercoordinates: if $\{e_i, r_\alpha : i = 1, \ldots, m, \alpha = 1, \ldots, n\}$ is a graded basis for $\mathbb{R}^m \oplus \mathbb{R}^n$ (so $|e_i| = 0$ and $|r_\alpha| = 1$) and $\{t^i, \vartheta^\alpha\}$ is the corresponding dual basis, then the set $\{t^i, \pi\vartheta^\alpha; \vartheta^\alpha, \pi t^i\}$ gives a supercoordinate system in $\mathbb{R}^{m+n|m+n}$. Here π is the natural morphism between V and ΠV.

2.2. The supertangent bundle

Our first aim is to describe the relation between the supertangent manifold as defined in [12, 4] and the supertangent bundle introduced by Sánchez-Valenzuela in [18].

If $\mathcal{M} = (M, \mathcal{A}_m)$ is a graded manifold of dimension (m, n), its supertangent bundle is defined via the one-to-one correspondence between equivalence classes of locally free sheaves of $\mathcal{A}_M$ modules over $\mathcal{M}$ of rank (r, s), and equivalence classes of supervector bundles over $\mathcal{M}$ of rank (r, s), considered as a natural generalization of the standard definition of vector bundles; namely, as the quadruplets $\{(E, \mathcal{A}_E), \Pi, (M, \mathcal{A}_M), V_S\}$ such that $\Pi : (E, \mathcal{A}_E) \to (M, \mathcal{A}_M)$ is a submersion of graded manifolds, V is a real (r, s)-dimensional supervector space and every $q \in M$ lies in a coordinate neighbourhood $\mathcal{U} \subseteq M$ for which an isomorphism, $\Psi_\mathcal{U}$, exists making the following diagram commutative:

$$\begin{array}{ccc} (\pi^{-1}(\mathcal{U}), \mathcal{A}_E(\pi^{-1}(\mathcal{U}))) & \xrightarrow{\Psi_\mathcal{U}} & (\mathcal{U}, \mathcal{A}_M(\mathcal{U})) \times V_S \\ \Pi \big\downarrow & & \big\downarrow P_1 \\ (\mathcal{U}, \mathcal{A}_M(\mathcal{U})) & = & (\mathcal{U}, \mathcal{A}_M(\mathcal{U})). \end{array} \tag{2.4}$$

In fact, the supertangent bundle is defined precisely as the supervector bundle of rank $(m, n) = \dim \mathcal{M}$ that corresponds to the sheaf of $\mathcal{A}_M$ modules Der $\mathcal{A}$.

As the superbundle $(E, \mathcal{A}_E)$ is locally isomorphic to a graded manifold of the form $(U, \mathcal{A}(\mathcal{U})) \times V_S$, we shall take advantage of this fact to describe the local supercoordinates of $(E, \mathcal{A}_E)$. Thus, if $\{q^i, \theta^\alpha\}$, $i = 1, \ldots, m$, $\alpha = 1, \ldots, n$, are local supercoordinates on $\mathcal{U} \subseteq M$, and $\{t^j, \pi\vartheta^\beta, \vartheta^\beta, \pi v^j\}$, $j = 1, \ldots, r$, $\beta = 1, \ldots, s$, are the local supercoordinates of $V_S = \mathbb{R}^{m+n|m+n}$ described previously, then $\{p_1^* q^i, p_2^* t^j, p_2^* \pi\vartheta^\beta, p_1^*\theta^\alpha, p_2^*\vartheta^\beta, p_2^*\pi t^j\}$, where $P_k = (p_k, p_k^*)$ is the natural projection of $(\mathcal{U}, \mathcal{A}(\mathcal{U})) \times V_S$ onto the kth factor, is a set of local supercoordinates on $(\mathcal{U}, \mathcal{A}(\mathcal{U})) \times V_S$, hence the image of this set under the morphism of superalgebras ψ^* will be a set of local supercoordinates for $(E, \mathcal{A}_E)$ on $\pi^{-1}(\mathcal{U})$, which, abusing the notation, we shall denote by $\{q^i, v^j, \pi\zeta^\beta, \theta^\alpha, \zeta^\beta, \pi v^j\}$.

Remark 2.1. We also want to point out that the superideal $\mathcal{I}$, locally generated by the superfunctions $\{\pi v^j, \pi\vartheta^\zeta\}$ ($1 \leqslant j \leqslant r$ and $1 \leqslant \beta \leqslant s$), defines a subsupermanifold of $(E, \mathcal{A}_E)$ of dimension $(m + r, n + s)$. Similarly, the superideal $\mathcal{I}'$ locally generated by the superfunctions $\{v^j, \zeta^\beta\}$ defines another subsupermanifold of $(E, \mathcal{A}_E)$ of dimension $(m + s, n + r)$.

2.3. Simple graded manifolds

Next we want to describe the supertangent bundle $ST\mathcal{M} := (STM, ST\mathcal{A})$ in a more concise way. With this in mind, we shall first make some comments on the Batchelor–Gawedzki structural theorem. Let $\pi : E \to M$ be a vector bundle of rank n, and $\bigwedge E$ its exterior algebra vector bundle (i.e. the vector bundle over M whose fibre on a point $q \in M$ is the vector space $\bigwedge E_q$). The sheaf of sections $\Gamma(\bigwedge E)$ can be considered, in the obvious way, as a sheaf of supercommutative superalgebras over M. Moreover, $(M, \Gamma(\bigwedge E))$ is a graded manifold. Indeed, if $\{(\mathcal{U}_k, \phi_k)\}$ is an atlas of M such that $\pi^{-1}(\mathcal{U}_k)$ trivialize E, then we have diffeomorphisms $\phi_k : \mathcal{U}_k \to U_k \subseteq \mathbb{R}^m$ and $\psi_k : \pi^{-1}(\mathcal{U}_k) \to \mathcal{U}_k \times \mathbb{R}^n$ such that $pr_1 \circ \psi_k = \pi|_{\mathcal{U}_k}$. Consider the superdomain $(U_k, \mathcal{A}_{m,n}(U_k))$ and let $\{u_k^i, \xi_k^\alpha\}$ ($i = 1, \ldots, m$ and $\alpha = 1, \ldots, n$), be supercoordinates on it. Now, if $\theta_k^\alpha : \mathcal{U}_k \to \pi^{-1}(\mathcal{U}_k)$ is the local section of $\bigwedge E$ defined by $\theta_k^\alpha(u) = \psi_k^{-1}(u, e_\alpha)$, where $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, it is clear that the morphism $\Phi_k : (\mathcal{U}_k, \Gamma(\bigwedge \pi^{-1}(\mathcal{U}_k)) \to (U_k, \mathcal{A}_{m,n}(U_k))$ defined by the assignments

$$u_k^i \mapsto q_k^i := \pi_i \circ \phi_k \qquad \text{and} \qquad \xi_k^\alpha \mapsto \theta_k^\alpha \tag{2.5}$$

where $\pi_i : \mathbb{R}^m \to \mathbb{R}$ is the projection onto the ith factor, is a chart, in the sense of graded manifolds, for $(M, \Gamma(\bigwedge E))$. Moreover, it is easy to check that, if $\mathcal{U}_{kl} := \mathcal{U}_k \cap \mathcal{U}_l \neq \emptyset$ the transition function of this graded manifold

$$\Phi_{kl} : (\phi_l(\mathcal{U}_{kl}), \mathcal{A}_{m,n}(\phi_l(\mathcal{U}_{kl}))) \to (\phi_k(\mathcal{U}_{kl}), \mathcal{A}_{m,n}(\phi_k(\mathcal{U}_{kl}))) \tag{2.6}$$

is given by the relations

$$\phi_{kl}^*(u_k^i) = \phi_{kl}^i \tag{2.7a}$$

$$\phi_{kl}^*(\xi_k^\alpha) = (\psi_{lk})_{\beta\alpha}\xi_l^\beta \tag{2.7b}$$

where $\phi_{kl} = \phi_k \circ \phi_l^{-1}$ denotes the change of coordinates in M, $\psi_{kl} = \psi_k \circ \psi_l^{-1}$ is the transition function of the vector bundle $\pi : E \to M$ over $\mathcal{U}_{kl}$, and $(\psi_{kl})_{\alpha\beta}$ is the matrix associated to ψ_{kl}. We refer to this kind of graded manifolds as simple graded manifolds.

Simple graded manifolds are more than just a nice example of graded manifolds. Indeed, it is not hard to obtain a fibre bundle out of a graded manifold. Let $\{\mathcal{U}_j\}$ be an open cover of M such that on each $\mathcal{U}_j$ one has local charts of $\mathcal{M}$, say $\Phi_j : (\mathcal{U}_j, \mathcal{A}_M(\mathcal{U}_j)) \to (U_j, \mathcal{A}_{m,n}(U_j))$ and let $\{U_j^i, \xi_j^\alpha\}$ ($i = 1, \ldots, m$ and $\alpha = 1, \ldots, n$) be supercoordinates on $(U_j, \mathcal{A}_{m,n}(U_j))$. If the transition morphisms are given by the relations

$$\phi_{jk}^*(u_j^i) = (\phi_{jk})_0^i(u) + (\phi_{jk})_{\alpha\beta}^i(u)\xi_k^\alpha\xi_k^\beta + \cdots \tag{2.8a}$$

$$\phi_{jk}^*(\xi_j^\alpha) = (\varphi_{jk})_\beta^\alpha(u)\xi_k^\beta + (\varphi_{jk})_{\beta\gamma\delta}^\alpha(u)\xi_k^\beta\xi_k^\gamma\xi_k^\delta + \cdots \tag{2.8b}$$

then, from the cocycle relations of the Φ_{jk}'s it follows that the matrices $(\varphi_{jk})_{\alpha\beta}$ satisfy, on each point of $\phi_k(\mathcal{U}_j \cap \mathcal{U}_k \cap \mathcal{U}_l)$, the cocycle relations

$$\varphi_{jk} \circ \varphi_{kl} = \varphi_{jl}. \tag{2.9}$$

Thus the functions $\tilde{\varphi}_{jk} : \mathcal{U}_j \cap \mathcal{U}_k \to \mathrm{GL}(n, \mathbb{R})$, defined by $\tilde{\varphi}_{jk}(q) = (\varphi_{jk}(\phi_k(q)))_{\alpha\beta}$, give rise to a vector bundle $E \to M$. Now, if we also assume that the $\mathcal{U}_j$'s are such that the $\pi^{-1}(\mathcal{U}_j)$'s trivialize $E \to M$, then, by our previous argument, we have a local chart $\Psi_j : (\mathcal{U}_j, \Gamma(\bigwedge \pi^{-1}(\mathcal{U}_j))) \to (U_j, \mathcal{A}_{m,n}(U_j))$ of $(M, \Gamma(\bigwedge E))$. Moreover, $\psi_j^* \circ (\phi_j^*)^{-1}$ is an isomorphism from the superalgebra $\mathcal{A}(\mathcal{U}_j)$ into the superalgebra $\Gamma(\bigwedge \pi^{-1}(\mathcal{U}_j))$. Thus, the graded manifolds $(M, \mathcal{A}_M)$ and $(M, \Gamma(\bigwedge E))$ are locally isomorphic. Suprisingly enough, these graded manifolds are globally isomorphic, although not in a canonical way, a fact known as the structural theorem of Batchelor [2] and Gawedzki [9].

Remark 2.2. What we want to emphasize is that, from (2.7), an explicit way to construct the so-called structural bundle $E \to M$ is to use the functions φ_{jk}, the first term of the second equation of (2.8), as the transition functions of the dual bundle E^*.

2.4. The underlying manifold of the supertangent bundle

In order to describe the tangent superbundle $ST\mathcal{M} := (STM, ST\mathcal{A})$ we shall follow the general construction of a supervector bundle out of a sheaf of $\mathcal{A}_M$ modules given in [18] applied to the sheaf of supervector fields Der $\mathcal{A}$. Let $\mathcal{U}$ be an open subset of M such that $(\mathcal{U}, \mathcal{A}_M(\mathcal{U}))$ is isomorphic to a superdomain; if $X = \sum_{i=1}^m X^i \partial_{q^i} + \sum_{\alpha=1}^n \chi^\alpha \partial_{\theta^\alpha}$ is a supervector field in Der $\mathcal{A}(\mathcal{U})$, then the map

$$g_\mathcal{U} : X \mapsto (X^{-1}, \ldots, X^m, \chi^1, \ldots, \chi^n) \tag{2.10}$$

defines an isomorphism between the sheaves of $\mathcal{A}_M$ modules $\mathcal{A}(\mathcal{U})^m \oplus \mathcal{A}(\mathcal{U})^n$ and Der $\mathcal{A}(\mathcal{U})$. Moreover, if $(\mathcal{U}_1, \mathcal{A}_M(\mathcal{U}_1))$ and $(\mathcal{U}_2, \mathcal{A}_M(\mathcal{U}_2))$ are two of such superdomains then the map

$$\begin{aligned} g_{12} = g_1(\mathcal{U}_1 \cap \mathcal{U}_2) \circ g_2^{-1}(\mathcal{U}_1 \cap \mathcal{U}_2) &: \mathcal{A}(\mathcal{U}_1 \cap \mathcal{U}_2)^m \oplus \mathcal{A}(\mathcal{U}_1 \cap \mathcal{U}_2)^n \\ &\to \mathcal{A}(\mathcal{U}_1 \cap \mathcal{U}_2)^m \oplus \mathcal{A}(\mathcal{U}_1 \cap \mathcal{U}_2)^n \end{aligned} \tag{2.11}$$

which basically expresses the change of supercoordinates of the supervector field X, is an isomorphism of $\mathcal{A}(\mathcal{U}_1 \cap \mathcal{U}_2)$ modules and is explicitly given by the graded matrix

$$g_{12} = \begin{pmatrix} A_{12} & \Theta_{12} \\ \Gamma_{12} & D_{12} \end{pmatrix} = \begin{pmatrix} \frac{\partial q_1^i}{\partial q_2^j} & \frac{\partial q_1^i}{\partial \theta_2^\beta} \\ \frac{\partial \theta_1^\alpha}{\partial q_2^j} & \frac{\partial \theta_1^\alpha}{\partial \theta_2^\beta} \end{pmatrix}. \tag{2.12}$$

Since g_{12} is invertible then the matrices $\tilde{A}_{12}$ and $\tilde{D}_{12}$, obtained from A_{12} and D_{12}, respectively, by projecting their entries onto $C^\infty(\mathcal{U}_1 \cap \mathcal{U}_2)$, are also invertible [14]; moreover, since the g's satisfy the cocycle condition, we also have

$$\tilde{A}_{12} \circ \tilde{A}_{23} = \tilde{A}_{13} \qquad \text{and} \qquad \tilde{D}_{12} \circ \tilde{D}_{23} = \tilde{D}_{13}. \tag{2.13}$$

The conclusion is that the matrices

$$\tilde{g}_{12} = \begin{pmatrix} \tilde{A}_{12} & 0 \\ 0 & \tilde{D}_{12} \end{pmatrix} \tag{2.14}$$

give rise to a smooth vector bundle $\tau : STM \to M$, which is the Whitney sum of the vector bundle determined by the transition functions $\tilde{A}_{12} = \frac{\partial q_1^i}{\partial q_2^j}$, which is nothing but the tangent bundle of the manifold M, and the vector bundle $\tilde{E} \to M$ determined by the $\tilde{D}$'s, which by remark 2.2, is isomorphic to the dual bundle of $\mathcal{M}$. Therefore, we have proved the following proposition:

Proposition 2.1. If $E \to M$ is a vector bundle such that $(M, \mathcal{A}) \cong (M, \Gamma \bigwedge(E))$, then the underlying manifold of the tangent superbundle of $\mathcal{M}$ is

$$STM = TM \oplus E^*. \tag{2.15}$$

2.5. The sheaf $ST\mathcal{A}$

To complete the description of the tangent superbundle we should describe the sheaf $ST\mathcal{A}$. This description is done in terms of the matrices (2.12) taking in consideration the fact that locally $ST\mathcal{M}$ is isomorphic to $(\mathcal{U}, \mathcal{A}(\mathcal{U})) \times \mathbb{R}^{m+n|m+n}$. Thus, if $\tau : TM \oplus E^* \to M$ is the canonical projection, then, according to [18], $ST\mathcal{A}$ is constructed using the superdomains $(\tau^{-1}(\mathcal{U}_j), ST\mathcal{A}(\tau^{-1}(\mathcal{U}_j)))$ and the superalgebra morphisms defined by the relations

$$q_1^i = \hat{\Phi}_{12}(q_1^i) = \phi_0^i(q) + \phi_{\alpha\beta}^i(q)\theta_2^\alpha \theta_2^\beta + \cdots \tag{2.16a}$$

$$\theta_1^\alpha = \hat{\Phi}_{12}(\theta_1^\alpha) = \psi_\beta^\alpha(q)\theta_2^\beta + \psi_{\beta\gamma\delta}^\alpha(q)\theta_2^\beta \theta_2^\gamma \theta_2^\delta + \cdots \tag{2.16b}$$

$$v_1^i = \hat{\Phi}_{12}(v_1^i) = \sum_{j=0}^{m} \frac{\partial q_1^i}{\partial q_2^j} v_2^j - \sum_{\beta=0}^{n} \frac{\partial q_1^i}{\partial \theta_2^\beta} \zeta_2^\beta$$

$$= \left(\frac{\partial \phi_0^i}{\partial q^j} + \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta_2^\alpha \theta_2^\beta + \cdots \right) v_2^j + (2\phi_{\alpha\beta}^i(q)\theta_2^\alpha + \cdots)\zeta_2^\beta \tag{2.16c}$$

$$\pi\zeta_1^\alpha = \hat{\Phi}_{12}(\pi\zeta_1^\alpha) = -\sum_{j=0}^{m} \frac{\partial \theta_1^\alpha}{\partial q_2^j} \pi v_2^j + \sum_{\beta=0}^{n} \frac{\partial \theta_1^\alpha}{\partial \theta_2^\beta} \pi \zeta_2^\beta$$

$$= -\left(\frac{\partial \psi_\beta^\alpha}{\partial q_2^j} \theta_2^\beta + \cdots \right) \pi v_2^j + (\psi_\beta^\alpha + 3\psi_{\beta\gamma\delta}^\alpha \theta_2^\gamma \theta_2^\delta + \cdots)\pi\zeta_2^\beta \tag{2.16d}$$

$$\zeta_1^\alpha = \hat{\Phi}_{12}(\zeta_1^\alpha) = \sum_{j=0}^{m} \frac{\partial \theta_1^\alpha}{\partial q_2^j} v_2^j + \sum_{\beta=0}^{n} \frac{\partial \theta_1^\alpha}{\partial \theta_2^\beta} \zeta_2^\beta$$

$$= \left(\frac{\partial \psi_\beta^\alpha}{\partial q_2^j} \theta_2^\beta + \cdots \right) v_2^j + (\psi_\beta^\alpha(q) + 3\psi_{\beta\gamma\delta}^\alpha(q) \theta_2^\gamma \theta_2^\delta + \cdots)\zeta_2^\beta \tag{2.16e}$$

$$\pi v_1^i = \hat{\Phi}_{12}(\pi v_1^i) = \sum_{j=0}^{m} \frac{\partial q_1^i}{\partial q_2^j} \pi v_2^j + \sum_{\beta=0}^{n} \frac{\partial q_1^i}{\partial \theta_2^\beta} \pi \zeta_2^\beta$$

$$= \left(\frac{\partial \phi_0^i}{\partial q^j} + \frac{\partial \phi_{\alpha\beta}^i}{\partial q^j} \theta_2^\alpha \theta_2^\beta + \cdots \right) \pi v_2^j + (-2\phi_{\alpha\beta}^i \theta_2^\alpha + \cdots)\pi\zeta_2^\beta \tag{2.16f}$$

where $\{q_j^i, v_j^i, \pi\zeta_j^\alpha, \theta_j^\alpha, \zeta_j^\alpha, \pi v_j^i\}$ are the supercoordinates on $\tau^{-1}(\mathcal{U}_j)$ described in section 2.2.

Now according to remark 2.2, the transition functions of the structural bundle $E' \to STM$ of $(STM, ST\mathcal{A})$ are obtained from (2.16); actually, they are the inverse transpose of the linear functions $\Psi_{12} : \tau^{-1}(\mathcal{U}_1) \cap \tau^{-1}(\mathcal{U}_2) \to \mathrm{GL}(2n+m, \mathbb{R})$ given by

$$\Psi_{12}(q, v, \pi\zeta) = \begin{pmatrix} \psi_\beta^\alpha(q) & 0 & 0 \\ \frac{\partial \psi_\beta^\alpha}{\partial q^i} v^i & \psi_\beta^\alpha(q) & 0 \\ -2\phi_{\alpha\beta}^i(q)\pi\zeta^\beta & 0 & \frac{\partial \phi_0^i}{\partial q^j} \end{pmatrix}. \tag{2.17}$$

Here $\{q, v, \pi\zeta\}$ are local coordinates on STM. Nevertheless, by our arguments in section 2.3 (i.e. the Batchelor–Gawedzki theorem) we may assume that $\phi_{\alpha\beta}^i(q) = 0$. Then, the following proposition follows immediately from (2.17).

Proposition 2.2. If $E \to M$ is a vector bundle such that $(M, \mathcal{A}) \cong (M, \Gamma \bigwedge(E))$, then the structural bundle of $ST\mathcal{M}$ is isomorphic to $(TE \oplus TM)^* \to TM \oplus E^*$.

We point out that, using different arguments, the tangent supermanifold has also been studied in [17].

Finally we notice that the subsupermanifold that corresponds to $ST\mathcal{M}$, according to remark 2.1, is nothing but the tangent supermanifold $(TM, T\mathcal{A})$ introduced by Ibort and Marín-Solano in [12].

2.6. Supervector fields as geometric sections

The main reason for considering the tangent superbundle $\{(STM, ST\mathcal{A}), \mathcal{T}, (M, \mathcal{A}), V_S)\}$, and supervector bundles in general [18], is that their geometric sections are in a one-to-one correspondence with the sections of the corresponding locally free sheaf of graded

$\mathcal{A}$ modules; in our case with the sections of the sheaf Der $\mathcal{A}$, in other words, with the supervector fields over $\mathcal{M}$. Following [18] we will make this correspondence explicit in the particular case we are interested in. The central point of this correspondence is to notice that both the geometric sections and the 'algebraic' sections, when restricted to an appropriate open set, are isomorphic to $\mathrm{Maps}((\mathcal{U}, \mathcal{A}(\mathcal{U})), V_S)$ the morphisms between the graded manifolds $(\mathcal{U}, \mathcal{A}(\mathcal{U}))$ and V_S. First of all, we notice that

$$\mathrm{Der}\mathcal{A}(\mathcal{U}) \cong \mathcal{A}(\mathcal{U})^m \oplus \mathcal{A}(\mathcal{U})^n \cong \mathrm{Maps}((\mathcal{U}, \mathcal{A}(\mathcal{U})), V_S). \tag{2.18}$$

If $X \in \mathrm{Der}\mathcal{A}(\mathcal{U})$ is written in local coordinates as $X = \sum_{i=1}^m X^i \partial_{q^i} + \sum_{\alpha=1}^n \chi^\alpha \partial_{\theta^\alpha}$, then (2.18) is implemented by the maps

$$X \mapsto (X^1, \ldots, X^m, \chi^1, \ldots, \chi^n) \mapsto \Phi_X \tag{2.19}$$

where $\Phi_X = (\phi_X, \phi_X^*) \in \mathrm{Maps}((\mathcal{U}, \mathcal{A}(\mathcal{U})), V_S)$ is the morphism described by, see [14], the morphism of superalgebras $\phi_X^* : \mathcal{A}_{m+n,m+n} \to \mathcal{A}(\mathcal{U})$ corresponding to the assignments:

$$\begin{array}{ll} t^i \mapsto X_0^i & \pi\vartheta^\alpha \mapsto \chi_0^\alpha \\ \vartheta^\alpha \mapsto \chi_1^\alpha & \pi t^i \mapsto X_1^i \end{array} \tag{2.20}$$

where X_0^i denote the even part of $X^i \in \mathcal{A}(U)$, and so on. On the other hand, if $ST\mathcal{A}(\mathcal{U})$ is a short notation for $ST\mathcal{A}(\tau^{-1}(\mathcal{U}))$ and $F = (f, f^*)$ is a morphism in $\mathrm{Maps}((\mathcal{U}, \mathcal{A}(\mathcal{U}), V_S)$, then $\Sigma_F : (\mathcal{U}, \mathcal{A}(\mathcal{U})) \to (\tilde{\tau}^{-1}(\mathcal{U}), ST\mathcal{A}(\mathcal{U}))$ will denote the section of the tangent superbundle described by the morphism of superalgebras $\sigma_F^* : ST\mathcal{A}(\mathcal{U}) \to \mathcal{A}(\mathcal{U})$ defined by the assignments

$$\begin{array}{ll} q^i \mapsto q^i & \theta^\alpha \mapsto \theta^\alpha \\ v^i \mapsto f^*(t^i) & \zeta^\alpha \mapsto f^*(\vartheta^\alpha) \\ \pi\zeta^\alpha \mapsto f^*(\pi\vartheta^\alpha) & \pi v^i \mapsto f^*(\pi t^i). \end{array} \tag{2.21}$$

(We remind the reader of our notation concerning the supercoordinates described in sections 2.1 and 2.2.) It is easy to check that

$$\mathrm{Der}\mathcal{A}(\mathcal{U}) \cong \mathrm{Maps}((\mathcal{U}, \mathcal{A}(\mathcal{U})), V_S) \cong \Gamma((\mathcal{U}, \mathcal{A}(\mathcal{U})), (\tilde{\tau}^{-1}(\mathcal{U}), ST\mathcal{A}(\mathcal{U}))) \tag{2.22}$$

is implemented by the morphisms:

$$X \mapsto \Phi_X \mapsto \Sigma_X \tag{2.23}$$

where $\sigma_X^* : ST\mathcal{A}(\mathcal{U}) \to \mathcal{A}(\mathcal{U})$ is given by the assignments

$$\begin{array}{ll} q^i \mapsto q^i & \theta^\alpha \mapsto \theta^\alpha \\ v^i \mapsto X_0^i & \zeta^\alpha \mapsto \chi_1^\alpha \\ \pi\zeta^\alpha \mapsto \chi_0^\alpha & \pi v^i \mapsto X_1^i. \end{array} \tag{2.24}$$

3. Supervector fields along a morphism

Since the information of a graded manifold is concentrated in the algebraic part, that is in the sheaf of superalgebras, to carry over the point constructions of the classical geometry in the graded context is somewhat difficult; for instance the notion of a flow of a supervector field is far from trivial [16, 10, 11]. To tackle these problems we introduced in [4] the notion of a supervector field along a morphism, which also turned out to be a useful tool to study (higher-order) supermechanics [5]. Nevertheless, there they were defined as some kind of superderivations, and our goal now is to give to such supervector fields a geometric description similar to the one in non-graded geometry. It is important to point out that, already in the non-graded context, vector fields along a map simplify several constructions [6–8].

3.1. Definition

Definition 3.1. Let $\Phi = (\phi, \phi^*) : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a morphism of graded manifolds; a homogeneous supervector field along Φ is a morphism of sheaves over M, $X : \mathcal{A} \to \Phi_* \mathcal{B}$ such that for each open subset $\mathcal{U}$ of M

$$X(fg) = X(f)\phi^*_{\mathcal{U}}(g) + (-1)^{|X||f|}\phi^*_{\mathcal{U}}(f)X(g) \tag{3.1}$$

whenever $f \in \mathcal{A}(\mathcal{U})$ is homogeneous of degree $|f|$. The sheaf of supervector fields along Φ will be denoted by $\mathfrak{X}(\Phi)$.

If X is a supervector field on $(M, \mathcal{A})$, then

$$\hat{X} := \phi^* \circ X \in \mathfrak{X}(\Phi) \tag{3.2}$$

is a supervector field along Φ. In particular, when Φ is a regular closed imbedding [14], $\hat{X}$ is nothing but the restriction of the supervector field X to the graded submanifold $\mathcal{N}$.

If Y is a supervector field on $(N, \mathcal{B})$, then

$$T\phi(Y) := Y \circ \phi^* \tag{3.3}$$

also belongs to $\mathfrak{X}(\Phi)$, and we say that Y is a projectable with respect to Φ if there exists $X \in \mathfrak{X}(\mathcal{A})$ such that

$$T\phi(Y) = \hat{X}. \tag{3.4}$$

$\mathfrak{X}(\Phi)$ is a locally free sheaf of $\Phi_*\mathcal{B}$ modules over M of rank $(m, n) = \dim \mathcal{M}$ [4]. Moreover, if $(q^i, \theta^\alpha)(1 \leqslant i \leqslant m, 1 \leqslant \alpha \leqslant n)$, are local supercoordinates on $\mathcal{U} \subset \mathcal{M}$, then

$$\partial_{\hat{q}^i} := \hat{\partial}_{q^i} \qquad \partial_{\hat{\theta}^\alpha} := \hat{\partial}_{\theta^\alpha} \tag{3.5}$$

form a local basis of $\mathfrak{X}(\Phi)(\mathcal{U})$. In particular, any $X \in \mathfrak{X}(\Phi)(\mathcal{U})$ can be written as

$$X = \sum_{i=1}^{m} X^i \partial_{\hat{q}^i} + \sum_{\alpha=1}^{n} \chi^\alpha \partial_{\hat{\theta}^\alpha} \tag{3.6}$$

where $X^i = X(q^i)$ and $\chi^\alpha = X(\theta^\alpha)$ are superfunctions in $\mathcal{B}(\phi^{-1}(\mathcal{U}))$ (denoted, from now on, by $\mathcal{B}(\mathcal{U})$ for short).

3.2. Supervector fields along a morphism as sections along a morphism

Geometrical sections of a supervector bundle are defined as usual.

Definition 3.2. Let $\Phi : (N, \mathcal{B}) \to (M, \mathcal{A})$ be a morphism of graded manifolds and let $\{(E, \mathcal{A}_E), \Pi, (M, \mathcal{A}_M), V_S\}$ be a supervector bundle over $\mathcal{M}$; a local section of $\mathcal{E} := (E, \mathcal{A}_E)$ along Φ over an open subset $\mathcal{U}$ of M is a morphism $\Sigma = (\sigma, \sigma^*) : (\phi^{-1}(\mathcal{U}), \mathcal{B}(\mathcal{U})) \to (\pi^{-1}(\mathcal{U}), \mathcal{A}_E(\mathcal{U}))$, where again $\mathcal{A}_E(\mathcal{U}) := \mathcal{A}_E(\pi^{-1}(\mathcal{U}))$, satisfying the condition

$$\Phi_{\mathcal{U}} = \Pi_{\mathcal{U}} \circ \Sigma_{\mathcal{U}}. \tag{3.7}$$

Here the subscript $\mathcal{U}$ means the restriction of the morphism to the corresponding open graded submanifold. The set of such sections will be denoted by $\Gamma_\Phi(\Pi|_{\mathcal{U}})$.

It is straightforward to check that the assignment

$$W \mapsto \Gamma_\Phi(\Pi|_W) \tag{3.8}$$

for each open set $W \subseteq \mathcal{U}$, makes $\Gamma_\Phi(\Pi|_\mathcal{U})$ into a sheaf of $\Phi_*\mathcal{B}$ modules. Moreover, if $\mathcal{U}$ is a trivializing neighbourhood of the supervector bundle $\mathcal{E}$, then it is not hard to obtain a one-to-one correspondence between $\Gamma_\Phi(\Pi|_\mathcal{U})$ and Maps $((\phi^{-1}(\mathcal{U}), \mathcal{B}(\mathcal{U})), V_S)$; in particular, one concludes that $\Gamma_\Phi(\Pi|_\mathcal{U})$ is locally free.

Remark 3.1. In the case when the morphism Φ is the projection Π of the supervector bundle, there is a conical section $\mathcal{C}$, to wit the identity morphism on $\mathcal{E}$. It turns out that several relevant objects are defined using this section.

We now turn our attention to the case when the supervector bundle is the tangent superbundle $ST\mathcal{M}$, in other words, to supervector fields. The correspondence between supervector fields along a morphism and sections along a morphism of the supervector bundle is carried out along the same lines as in the case of the usual supervector fields (see section 2.6). Thus, one has

$$\mathfrak{X}(\Phi)(\mathcal{U}) \cong \mathcal{B}(\mathcal{U})^m \oplus \mathcal{B}(\mathcal{U})^n \cong \text{Maps}((\phi^{-1}(\mathcal{U}), \mathcal{B}(\mathcal{U})), V_S). \tag{3.9}$$

This correspondence is also implemented by (2.19), where now the superfunctions X^i and χ^α are given by (3.6). On the other hand, if $\mathcal{U}$ is also a trivializing neighbourhood of the supervector bundle $ST\mathcal{M}$, as before, one can check that

$$\text{Maps}((\phi^{-1}(\mathcal{U}), \mathcal{B}(\mathcal{U})), V_S) \cong \Gamma_\Phi(\mathcal{T}|_\mathcal{U}). \tag{3.10}$$

The explicit correspondence between a supervector field $X \in \mathfrak{X}(\Phi)(\mathcal{U})$ and a local section along Φ is given by

$$X \mapsto \Sigma_X \tag{3.11}$$

where $\sigma_X^* : ST\mathcal{A}(\mathcal{U}) \to \mathcal{B}(\mathcal{U})$ is defined by the assignments

$$\begin{aligned} &q^i \mapsto \phi^*(q^i) \qquad \theta^\alpha \mapsto \phi^*(\theta^\alpha) \\ &v^i \mapsto X_0^i \qquad \zeta^\alpha \mapsto \chi_1^\alpha \\ &\pi\zeta^\alpha \mapsto \chi_0^\alpha \qquad \pi v^i \mapsto X_1^i. \end{aligned} \tag{3.12}$$

3.3. *The total time derivative operator*

As in the non-graded context, the geometry of the tangent supermanifold is concentrated in two objects: the vertical superendomorphism and the total time derivative operator. Moreover this operator, introduced in [4], turned out to be quite important in the Lagrangian formalism of supermechanics. In what follows, we shall use the previous ideas to provide an intrinsic definition of the total time derivative operator.

Definition 3.3. The canonical section of the tangent supervector bundle $(ST\mathcal{M}, \mathcal{T}, \mathcal{M})$ along $\mathcal{T}$ described in remark 3.1, will be called the total time derivative operator and will be denoted by $\boldsymbol{T}$.

Since $\boldsymbol{T}$ is nothing but the identity morphism, according to the previous section $\boldsymbol{T}$ corresponds with the superderivation along $\mathcal{T}$ given, in terms of the standard supercoordinates of $ST\mathcal{M}$, by

$$\boldsymbol{T} = \sum_{i=1}^{m}(v^i + \pi v^i)\partial_{\hat{q}^i} + \sum_{\alpha=1}^{n}(\zeta^\alpha + \pi\zeta^\alpha)\partial_{\hat{\theta}^\alpha}. \tag{3.13}$$

As we shall see later, sometimes it is convenient to work with the tangent supermanifold $T\mathcal{M}$. Thus, if $\Phi : T\mathcal{M} \to ST\mathcal{M}$ is the regular closed imbedding that defines $T\mathcal{M}$ and that is locally defined by the relations

$$\begin{aligned} q^i &\mapsto q^i & \theta^\alpha &\mapsto \theta^\alpha \\ v^i &\mapsto v^i & \zeta^\alpha &\mapsto \zeta^\alpha \\ \pi\zeta^\alpha &\mapsto 0 & \pi v^i &\mapsto 0 \end{aligned} \tag{3.14}$$

then the restriction of $\boldsymbol{T}$ to $T\mathcal{M}$ would be the superderivation along the restriction of $\mathcal{T}$ to $T\mathcal{M}$ given by $\phi^* \circ \boldsymbol{T}$, and its local expression would be

$$\boldsymbol{T} = \sum_{i=1}^{m} v^i \partial_{\hat{q}^i} + \sum_{\alpha=1}^{n} \zeta^\alpha \partial_{\hat{\theta}^\alpha} \tag{3.15}$$

where now

$$\partial_{\hat{q}^i} = \phi^* \circ \tau^* \circ \partial_{q^i} \qquad \text{and} \qquad \partial_{\hat{\theta}^\alpha} = \phi^* \circ \tau^* \circ \partial_{\theta^\alpha}. \tag{3.16}$$

We shall make no distinction in the notation when we regard $\boldsymbol{T}$ as an operator either on $ST\mathcal{M}$ or on $T\mathcal{M}$.

4. Graded 1-forms along a morphism of supermanifolds

4.1. The cotangent superbundle and the cotangent supermanifold

The sheaf of graded 1-forms is, by definition, the dual sheaf of Der $\mathcal{A}$, and corresponds, according to [18], to a supervector bundle $(ST^*\mathcal{M}, \Pi, \mathcal{M}, V_S)$ that will be called the cotangent superbundle of $\mathcal{M}$. As one might expect, most of the ideas of the previous sections can be used with this sheaf of $\mathcal{A}$ modulos, taking into consideration what happens in the non-graded context.

Obviously $\Omega^1(\mathcal{A}) = \mathfrak{X}(\mathcal{A}^*)$ is locally free. Moreover, if $\mathcal{U}$ is an open subset of M, and $\{q^i, \theta^\alpha\}$ are local supercoordinates on it, then $\{\mathrm{d}q^1, \dots, \mathrm{d}q^m, -\mathrm{d}\theta^1, \dots, -\mathrm{d}\theta^n\}$ is the basis of the module $\Omega^1\mathcal{A}(\mathcal{U}) = (\mathrm{Der}\ \mathcal{A}(\mathcal{U}))^*$ dual to the basis $\{\partial_{q^i}, \partial_{\theta^\alpha}\}$ of $\mathfrak{X}(\mathcal{A}(\mathcal{U}))$. In particular, any $\omega \in \Omega^1\mathcal{A}(\mathcal{U})$ can be written in a unique way, in the form

$$\omega = \sum_{i=1}^{m} w^i\, \mathrm{d}q^i + \sum_{\alpha=1}^{n} \omega^\alpha\, \mathrm{d}\theta^\alpha \tag{4.1}$$

where the superfunctions w^i and ω^α are given by

$$w^i = \omega(\partial_{q^i}) \qquad \text{and} \qquad \omega^\alpha = -\omega(\partial_{\theta^\alpha}). \tag{4.2}$$

Naturally, one can describe the cotangent superbundle in a similar way as we described the tangent superbundle in sections 2.4 and 2.5, but, in analogy with the non-graded geometry, using instead the matrices $(g^{st}_{\alpha\beta})^{-1}$, where $g_{\alpha\beta}$ are the transition functions for the tangent superbundle (2.12) and st denote the supertranspose matrix.

The correspondence between the sections of the cotangent superbundle $ST^*\mathcal{M} = (ST^*M, ST^*\mathcal{A})$ and graded 1-forms is accomplished using the same ideas as in section 2.6. Thus, if in addition, the open subset $\mathcal{U}$ is a trivializing neighbourhood for $ST^*\mathcal{M}$, such that $(\pi^{-1}(\mathcal{U}), ST^*\mathcal{A}(\mathcal{U}))$ is also isomorphic to a superdomain, where $\Pi = (\pi, \pi^*)$ is the natural projection of $ST^*\mathcal{M}$ on $\mathcal{M}$, and $ST^*\mathcal{A}(\mathcal{U})$ is a short notation for $ST^*\mathcal{A}(\pi^{-1}(\mathcal{U}))$, then the correspondence

$$\Omega^1\mathcal{A}(\mathcal{U}) \cong \Gamma((\mathcal{U}, \mathcal{A}(\mathcal{U})), (\pi^{-1}(\mathcal{U}), ST^*\mathcal{A}(\mathcal{U}))) \tag{4.3}$$

is implemented by the morphism:

$$\omega \mapsto \Sigma_\omega \tag{4.4}$$

where $\sigma_\omega^* : ST^*\mathcal{A}(\mathcal{U}) \to \mathcal{A}(\mathcal{U})$ is defined by the assignments

$$\begin{aligned} &q^i \mapsto q^i \qquad \theta^\alpha \mapsto \theta^\alpha \\ &p^i \mapsto w_0^i \qquad \eta^\alpha \mapsto \omega_1^\alpha \\ &\pi\eta^\alpha \mapsto \omega_0^\alpha \qquad \pi p^i \mapsto w_1^i \end{aligned} \tag{4.5}$$

where the w^i and the ω^α are as in (4.2) and the subindices 0 or 1 stand for the even or odd components. Once more, we remind the reader of our notation concerning local supercoordinates of supervector bundles.

In analogy with the tangent superbundle, the subsupermanifold $T^*\mathcal{M} = (T^*M, T^*\mathcal{A})$ of $ST^*\mathcal{M}$, of dimension $(2m, 2n)$, associated to the superideal $\mathcal{I}^*$ locally generated by the superfunctions $\{\pi p^i, \pi\eta^\alpha\}$ will be called the cotangent supermanifold.

4.2. Graded forms along a morphism

Definition 4.1. Let $\Phi : (N, \mathcal{B}) \mapsto (M, \mathcal{A})$ be a morphism of graded manifolds; we define $\Omega^1(\Phi)$, the sheaf of graded 1-forms along Φ, as the sheaf of $\phi_*\mathcal{B}$ modules dual to the sheaf $\mathfrak{X}(\Phi)$. In other words,

$$\Omega^1(\Phi) = \mathfrak{X}(\Phi)^* = \mathrm{Hom}(\mathfrak{X}(\Phi), \phi_*\mathcal{B}). \tag{4.6}$$

In general, k-superforms are defined as

$$\Omega^k(\Phi) := \bigwedge^k(\Omega^1(\Phi)) \tag{4.7}$$

where the wedge product is to be understood in the sense of graded algebras.

Since $\Omega^1(\Phi)$ is the dual of a locally free $\phi_*\mathcal{B}$ modulo, is itself a locally free $\phi_*\mathcal{B}$ modulo. Moreover, if ω is a graded 1-form on $\mathcal{M}$, the restriction of ω to $\mathcal{N}$ is the graded 1-form along Φ defined by

$$\hat{\omega}(\hat{X}) := \phi^* \circ \omega(X) \qquad \forall X \in \mathfrak{X}(\mathcal{A}_M). \tag{4.8}$$

If (q^i, θ^α) are supercoordinates of $\mathcal{M}$ on $\mathcal{U}$, and $\mathrm{d}\hat{q}^i, \mathrm{d}\hat{\theta}^\alpha$ are the restrictions of $\mathrm{d}q^i$ and $\mathrm{d}\theta^\alpha$ respectively, then

$$\mathrm{d}\hat{q}^i(\partial_{\hat{q}_i}) = \delta_{ij} \qquad \mathrm{d}\hat{q}^i(\partial_{\hat{\theta}^\beta}) = 0 \qquad \mathrm{d}\hat{\theta}^\alpha(\partial_{\hat{q}^j}) = 0 \qquad \mathrm{d}\hat{\theta}^\alpha(\partial_{\hat{\theta}^\beta}) = -\delta_{\alpha\beta} \tag{4.9}$$

hence $\{\mathrm{d}\hat{q}^i, -\mathrm{d}\hat{\theta}^\alpha\}$ is the dual basis of $\{\partial_{\hat{q}^i}, \partial_{\hat{\theta}^\alpha}\}$. In particular, any graded 1-form ω along Φ can be written locally as

$$\omega = \sum_{i=1}^m w^i\,\mathrm{d}\hat{q}^i + \sum_{\alpha=1}^n \omega^\alpha\,\mathrm{d}\hat{\theta}^\alpha \tag{4.10}$$

where the superfunctions w^i and ω^α belong to $\mathcal{B}(\mathcal{U})$, and are defined by

$$w^i = \omega(\partial_{\hat{q}^i}) \qquad \text{and} \qquad \omega^\alpha = -\omega(\partial_{\hat{\theta}^\alpha}). \tag{4.11}$$

The equivalent process to (3.3) does not work here; instead, if ω is a graded 1-form along Φ, then $\phi^\sharp\omega$ given by

$$\phi^\sharp\omega(Y) := \omega(T\phi(Y)) \qquad \forall Y \in \mathfrak{X}(\mathcal{B}) \tag{4.12}$$

is a graded 1-form on $\mathcal{N}$. As a matter of fact, it is possible to classify the graded 1-forms on $\mathcal{N}$ that come from graded 1-forms along Φ, when Φ is a submersion. The result is that $\Omega^1(\Phi)$ is isomorphic to the $\phi_*\mathcal{B}$ modulo of Φ semibasic 1-forms on $\mathcal{N}$ [4].

Naturally, this construction, together with the last result, can be generalized to graded k-forms. For instance, if $\omega \in \Omega^k(\Phi)$, then

$$\phi^\sharp\omega(Y_1,\ldots,Y_k) := \omega(T\phi(Y_1),\ldots,T\phi(Y_k)). \tag{4.13}$$

The important point is that these two processes can be combined to give an intrinsic definition of the pull back of a graded form; something that, to our knowledge, was lacking in the graded context.

Definition 4.2. Let $\Phi : (N,\mathcal{B}) \to (M,\mathcal{A})$ be a morphism of graded manifolds and let μ be a graded k-form on $\mathcal{M}$. The pull back of μ by Φ is the graded k-form on $\mathcal{N}$ given by

$$\Phi^*(\mu) := \phi^\sharp(\hat{\mu}). \tag{4.14}$$

If μ is the graded 1-form given in local supercoordinates by $\mu = \sum_{i=1}^m u^i\,\mathrm{d}q^i + \sum_{\alpha=1}^n \mu^\alpha\,\mathrm{d}\theta^\alpha$, then

$$\hat{\mu} = \sum_{i=1}^m \phi^*(u^i)\,\mathrm{d}\hat{q}^i + \sum_{\alpha=1}^n \phi^*(\mu^\alpha)\mathrm{d}\hat{\theta}^\alpha \tag{4.15}$$

on the other hand, if $Y \in \mathfrak{X}(\mathcal{N})$ is given in local coordinates by $Y = \sum_{j=1}^r Y^j\partial_{p^j} + \sum_{\beta=1}^s \Upsilon^\beta\partial_{\eta^\beta}$, and $\phi^i := \phi^*(q^i)$ and $\phi^\alpha := \phi^*(\theta^\alpha)$ are the coordinate representation of Φ [14], then

$$\begin{aligned}(\Phi^*\mu)(Y) = {}& \sum_{ij}\phi^*(u^i)Y^j\frac{\partial\phi^i}{\partial p^j} + \sum_{i\beta}\phi^*(u^i)\Upsilon^\beta\frac{\partial\phi^i}{\partial\eta^\beta}\\ &+(-1)^{|Y|}\sum_{j\alpha}\phi^*(\mu^\alpha)Y^j\frac{\partial\phi^\alpha}{\partial p^j} + (-1)^{|Y|}\sum_{\alpha\beta}\phi^*(\mu^\alpha)\Upsilon^\beta\frac{\partial\phi^\alpha}{\partial\eta^\beta}\end{aligned} \tag{4.16}$$

which is the definition given [13].

The following technical result will be needed later on.

Lemma 4.1. Let $\Phi = (\phi,\phi^*) : (N,\mathcal{B}) \to (M,\mathcal{A})$ be a diffeomorphism, and μ a graded k-form on $\mathcal{M}$, then

$$\phi^{-1*}(\Phi^*\mu(Y_1,\ldots,Y_k)) = \mu(\phi^{-1*}\circ Y_1\circ\phi^*,\ldots,\phi^{-1*}\circ Y_k\circ\phi^*)). \tag{4.17}$$

Proof. Since Φ is a diffeomorphism any supervector field on $\mathcal{N}$ is projectable with respect to Φ; hence for each Y_i there exists $X_i \in \mathfrak{X}(\mathcal{A})$ on $\mathcal{M}$ such that $Y_i\circ\phi^* = \phi^*\circ X_i$, and one has

$$\begin{aligned}(\Phi^*\omega)(Y_1,\ldots,Y_k) &= (\phi^\sharp(\hat{\omega}))(Y_1,\ldots,Y_k) = \hat{\omega}(Y_1\circ\phi^*,\ldots,Y_k\circ\phi^*)\\ &= \hat{\omega}(\phi^*\circ X_1,\ldots,\phi^*\circ X_k) = \phi^*(\omega(X_1,\ldots,X_k))\end{aligned} \tag{4.18}$$

and since $X_i = \phi^{-1*}\circ Y\circ\phi^*$ the lemma follows. □

4.3. The canonical graded forms on the cotangent supervector bundle

As expected, graded 1-forms along a morphism have their geometric counterpart. If $\Phi : \mathcal{N} \to \mathcal{M}$ is a morphism of graded manifolds and $\mathcal{U} \subseteq \mathcal{M}$ is an open subset such that $(\mathcal{U},\mathcal{A}(\mathcal{U}))$ is isomorphic to a superdomain and trivialize the cotangent superbundle $\Pi : ST^*\mathcal{M} \to \mathcal{M}$, then the correspondence

$$\Omega^1(\Phi)(\mathcal{U}) \cong \Gamma_\Phi(\Pi|_\mathcal{U}) \tag{4.19}$$

is carried out using similar arguments as before and is given by

$$\omega \mapsto \Sigma_\omega \tag{4.20}$$

where $\sigma_\omega^* : ST^*\mathcal{A}(\mathcal{U}) \to \mathcal{B}(\mathcal{U})$ is defined by the assignments

$$\begin{aligned} &q^i \mapsto \phi^*(q^i) \qquad \theta^\alpha \mapsto \phi^*(\theta^\alpha) \\ &p^i \mapsto w_0^i \qquad \eta^\alpha \mapsto \omega_1^\alpha \\ &\pi\eta^\alpha \mapsto \omega_0^\alpha \qquad \pi p^i \mapsto w_1^i. \end{aligned} \tag{4.21}$$

Here w^i and ω^α are the superfunctions defined in (4.11), and the subindices 0 and 1 stand for the even and odd components, respectively.

Once again, when $\Phi = \Pi = (\pi, \pi^*)$ is the canonical projection of $ST^*\mathcal{M}$ on $\mathcal{M}$, we have, according to remark 3.1, a canonical section along Π, which, in view of (4.21), corresponds to the graded 1-form $\check{\Theta}_0$ in $\Omega^1(\Phi)$ locally given by

$$\check{\Theta}_0 = \sum_{i=1}^{m}(p^i + \pi p^i)\mathrm{d}\hat{q}^i + \sum_{\alpha=1}^{n}(\eta^\alpha + \pi\eta^\alpha)\mathrm{d}\hat{\theta}^\alpha. \tag{4.22}$$

Definition 4.3. The graded 1-form Π semibasic that corresponds to $\check{\Theta}_0 \in \Omega^1(\Phi)$ will be denoted by Θ_0, and we will refer to it as the canonical Liouville 1-form on $ST^*\mathcal{M}$.

From (4.22) it follows that (see [4])

$$\Theta_0 = \sum_{i=1}^{m}(p^i + \pi p^i)\mathrm{d}q^i + \sum_{\alpha=1}^{n}(\eta^\alpha + \pi\eta^\alpha)\mathrm{d}\theta^\alpha. \tag{4.23}$$

On the other hand, if $\Psi : T^*\mathcal{M} \to ST^*\mathcal{M}$ is the canonical closed imbedding of $T^*\mathcal{M}$ which locally is given by

$$\begin{aligned} &q^i \mapsto q^i \qquad \theta^\alpha \mapsto \theta^\alpha \\ &p^i \mapsto p^i \qquad \eta^\alpha \mapsto \eta^\alpha \\ &\pi\eta^\alpha \mapsto 0 \qquad \pi p^i \mapsto 0 \end{aligned} \tag{4.24}$$

then the restriction of Θ_0 to $T^*\mathcal{M}$, that will also be denoted by Θ_0, is locally given by

$$\Theta_0 = \sum_{i=1}^{m} p^i\,\mathrm{d}q^i + \sum_{\alpha=1}^{n} \eta^\alpha\,\mathrm{d}\theta^\alpha \tag{4.25}$$

where, to be precise $\mathrm{d}q^i$ and $\mathrm{d}\theta^\alpha$ stand for $\mathrm{d}(\phi^*(q^i))$ and $\mathrm{d}(\phi^*(\theta\alpha))$, respectively.

The canonical Liouville 1-form was defined in [18] in a different way and is equivalent to ours.

Theorem 4.1. The canonical Liouville 1-form Θ_0 is the only Π semibasic 1-form on $ST^*\mathcal{M}$ that satisfy

$$\Sigma_\omega^*(\Theta_0) = \omega \qquad \forall\omega \in \Omega^1(\mathcal{A}) \tag{4.26}$$

where Σ_ω is the section of the cotangent superbundle corresponding to ω.

Proof. It is enough to work on a local chart of $\mathcal{M}$. Thus, if $\omega = \sum_{i=1}^{m} w^i\,\mathrm{d}q^i + \sum_{\alpha=1}^{n} \omega^\alpha \mathrm{d}\theta^\alpha$ of an open subset $\mathcal{U}$ of M, we have

$$\Sigma_\omega^*(\Theta_0) = \sigma_\omega^\sharp(\hat\Theta_0) = \sigma_\omega^\sharp\left(\sum_{i=1}^{m}\sigma_\omega^*(p^i+\pi p^i)\mathrm{d}\hat q^i + \sum_{\alpha=1}^{n}\sigma_\omega^*(\eta^\alpha+\pi\eta^\alpha)\mathrm{d}\hat\theta^\alpha\right)$$
$$= \sigma_\omega^\sharp\left(\sum_{i=1}^{m} w^i\,\mathrm{d}\hat q^i + \sum_{\alpha=1}^{n}\omega^\alpha\,\mathrm{d}\hat\theta^\alpha\right) = \sum_{i=1}^{m} w^i\,\mathrm{d}(\sigma_\omega^*(q^i))$$
$$+\sum_{\alpha=1}^{n}\omega^\alpha \mathrm{d}(\sigma_\omega^*(\theta^\alpha)) = \sum_{i=1}^{m} w^i\,\mathrm{d}q^i + \sum_{\alpha=1}^{n}\omega^\alpha\mathrm{d}\theta^\alpha = \omega. \tag{4.27}$$

On the other hand, a general graded 1-form Θ on $ST^*\mathcal{M}$ is written locally as

$$\Theta = \sum_{i=1}^{m} A^i\,\mathrm{d}q^i + \sum_{i=1}^{m} B^i\,\mathrm{d}p^i + \sum_{\alpha=1}^{n} C^\alpha\,\mathrm{d}\pi\eta^\alpha + \sum_{\alpha=1}^{n} D^\alpha\,\mathrm{d}\theta^\alpha + \sum_{\alpha=1}^{n} E^\alpha\,\mathrm{d}\eta^\alpha + \sum_{i=1}^{m} F^i\,\mathrm{d}\pi p^i \tag{4.28}$$

but, if it is Π semibasic then B^i, C^α, E^α and F^i vanish, and the previous argument fixes the other two supercoordinates, and the uniqueness follows. □

Remark 4.1. Although Θ_0 is formally equal to the canonical 1-form of the cotangent bundle in non-graded geometry, it turns out that the graded 2-form $-\mathrm{d}\Theta_0$ is degenerate; nevertheless, if one restricts Θ_0 to the cotangent supermanifold $T^*\mathcal{M}$, then $-\mathrm{d}\Theta_0$ is a non-degenerate graded 2-form that will be called the canonical graded 2-form and will be denoted by Ω_0. We refer to [18] for details.

5. The super-Legendre transformation

5.1. The vertical superendomorphism

As in the non-graded case, in order to define intrinsically the vertical superendomorphism, we need to define vertical lifts. We shall accomplish this generalizing the ideas of the non-graded case (see, for instance [22]).

Let $\mathcal{U}$ be an open subset of M such that $(\mathcal{U}, \mathcal{A}(\mathcal{U}))$ is isomorphic to a superdomain. We associate to each superfunction $f \in \mathcal{A}(\mathcal{U})$ the superfunction $f^V \in T\mathcal{A}(\mathcal{U})$ defined by

$$f^V := \sum_{i=1}^{m}\frac{\partial F}{\partial q^i}v^i + \sum_{\alpha=1}^{n}\frac{\partial F}{\partial\theta^\alpha}\zeta^\alpha \tag{5.1}$$

where $F := \tau^*(f) \in T\mathcal{A}(\mathcal{U})$. It turns out that, any supervector field Y on $T\mathcal{M}$ is determined by its action on the superfunctions f^V:

Lemma 5.1. If $Y \in \mathfrak{X}(T\mathcal{A})$ satisfy

$$Y(f^V) = 0 \qquad \forall f \in \mathcal{A}(\mathcal{U}) \tag{5.2}$$

then $Y \equiv 0$ on $\tau^{-1}(\mathcal{U})$.

Proof. If the local expression for Y is

$$Y = \sum_{k=1}^{m} A^k\partial_{q^k} + \sum_{k=1}^{m} B^k\partial_{v^k} + \sum_{\gamma=1}^{n} C^\gamma\partial_{\theta^\gamma} + \sum_{\gamma=1}^{n} D^\gamma\partial_{\zeta^\gamma} \tag{5.3}$$

then

$$0 = Y(f^V) = \sum_{k,i}^{m} A^k \frac{\partial^2 F}{\partial q^k \partial q^i} v^i + \sum_{k,\alpha}^{m} A^k \frac{\partial^2 F}{\partial q^k \partial \theta^\alpha} \zeta^\alpha + \sum_{k=1}^{m} B^k \frac{\partial F}{\partial v^k}$$
$$+ \sum_{\gamma,i}^{m} C^\gamma \frac{\partial^2 F}{\partial \theta \gamma \partial q^i} q^i + \sum_{\gamma,\alpha}^{m} C^\gamma \frac{\partial^2 F}{\partial \theta^\gamma \partial \theta^\alpha} \zeta^\alpha + \sum_{\gamma=1}^{n} D^\gamma \frac{\partial F}{\partial \zeta^\gamma}. \tag{5.4}$$

Plugging $f = q^j$ in (5.4), one gets $B^j = 0$; similarly, if $f = \theta^\beta$ it follows that $D^\beta = 0$.

On the other hand, taking $f = q^l q^j$ in (5.4), one gets

$$A^l v^j + A^j v^l = 0 \tag{5.5}$$

in particular, if $l = j$, then $A^j v^j = 0$, and therefore $A^j = 0$. Similarly, using $f = q^j \theta^\beta$ one gets $C^\beta = 0$, and the lemma is proved. □

Definition 5.1. If X is a supervector field on $\mathcal{M}$, its vertical lift is the supervector field X^V on $T\mathcal{M}$ defined by

$$X^V(f^V) = \tau^*(X(f)) \qquad \forall f \in \mathcal{A}. \tag{5.6}$$

Similarly, if X is a supervector field along $\mathcal{T}$, then we define its vertical lift by the relations

$$X^V(f^V) = X(f) \qquad \forall f \in \mathcal{A}(\mathcal{U}). \tag{5.7}$$

In local supercoordinates, if $X = \sum_{i=1}^m X^i \partial_{\hat{q}^i} + \sum_{\alpha=1}^n \chi^\alpha \partial_{\hat{\theta}^\alpha}$, then

$$X^V = \sum_{i=1}^{m} X^i \partial_{v^i} + \sum_{\alpha=1}^{n} \chi^\alpha \partial_{\zeta^\alpha}. \tag{5.8}$$

The situation is slightly different in the tangent superbundle $ST\mathcal{M}$. The natural thing to do is to replace f^V by the superfunctions

$$f^V := \sum_{i=1}^{m} \frac{\partial F}{\partial q^i}(v^i + \pi v^i) + \sum_{\alpha=1}^{n} \frac{\partial F}{\partial \theta^\alpha}(\zeta^\alpha + \pi \zeta^\alpha). \tag{5.9}$$

Even though a general supervector field is not determined by its action on these superfunctions (for instance, $Y(f^V) = 0$ for all $f \in \mathcal{A}(\mathcal{U})$ of $Y = \partial_{v^i} - \partial_{\pi v^i}$), one can check, using the same argument as before, that homogeneous supervector fields are determined by its action on superfunctions of the form (5.9).

Thus we define the vertical lift of a homogeneous supervector field $X \in \mathfrak{X}(\mathcal{A})$ as the supervector field $X^V \in \mathfrak{X}(ST\mathcal{A})$ that satisfies

$$X^V(f^V) = \tau_0^*(X(f)) \qquad \forall f \in \mathcal{A}. \tag{5.10}$$

Moreover, if $X = X_0 + X_1$, then we define $X^V := X_0^V + X_1^V$.

Similarly, if X is a homogeneous supervector field along the canonical projection of $ST\mathcal{M}$ onto $\mathcal{M}$, its vertical lift is also defined by the equation (5.7), where now $\mathcal{T}$ denotes the projection of $ST\mathcal{M}$, and, of course, in the general case by $X^V := X_0^V + X_1^V$.

We are now in a position to define, in an intrinsic way, the two objects that encode all the geometric information of the tangent superbundle.

Definition 5.2. The vertical superendomorphism is the graded tensor field of type $(1, 1)$ $S : \mathfrak{X}(ST\mathcal{A}) \to \mathfrak{X}(ST\mathcal{A})$ defined by

$$S(Y) := T\tau(Y)^V. \tag{5.11}$$

The morphism of $T\mathcal{A}$ modulos $S : \mathfrak{X}(T\mathcal{M}) \to \mathfrak{X}(T\mathcal{M})$, defined also by (5.11), except that $\mathcal{T}$ now denotes the restriction to $T\mathcal{M}$, is also called vertical superendomorphism.

On the other hand, if

$$Y = \sum_{i=1}^{m} Y^i \partial_{q^i} + \sum_{i=1}^{m} \mathcal{Y}^i \partial_{v^i} + \sum_{\alpha=1}^{n} \tilde{\Xi}^\alpha \partial_{\pi\zeta^\alpha} + \sum_{\alpha=1}^{n} \Upsilon^\alpha \partial_{\theta^\alpha} + \sum_{\alpha=1}^{n} \Xi^\alpha \partial_{\zeta^\alpha} + \sum_{i=1}^{m} \tilde{\mathcal{Y}}^i \partial_{\pi v^i} \tag{5.12}$$

then, using the change rule [14],

$$SY = \sum_{i=1}^{m} Y^i \partial_{v^i} + \sum_{\alpha=1}^{n} \Upsilon^\alpha \partial_{\zeta^\alpha} + \sum_{i=1}^{m} Y^i \partial_{\pi v^i} + \sum_{\alpha=1}^{n} \Upsilon^\alpha \partial_{\pi\theta^\alpha}. \tag{5.13}$$

In particular, it is clear that

$$\operatorname{Im} S = \ker S = \{Y : Y \text{ is vertical with respect to } \mathcal{T}_0\} \tag{5.14}$$

and that the matrix of S, in terms of the supercoordinates we have been using, is

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ I & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5.15}$$

while the corresponding matrix for the vertical superendomorphism of $T\mathcal{M}$ would be

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix}. \tag{5.16}$$

Definition 5.3. The Liouville supervector field, Δ, is the vertical lift of the total time derivative. In other words, Δ is the supervector field on $\mathfrak{X}(ST\mathcal{A})$ (or $\mathfrak{X}(T\mathcal{A})$) defined by

$$\Delta = \boldsymbol{T}^V \tag{5.17}$$

5.2. Graded Cartan forms

In analogy with ordinary Lagrangian mechanics, the Cartan graded 1-form associated to a given Lagrangian superfunction L and $ST\mathcal{A}$ is defined by

$$\Theta_L := \mathrm{d}L \circ S. \tag{5.18}$$

Using (5.13) it is easy to check that in local supercoordinates

$$\Theta_L = \left(\frac{\partial L}{\partial v^i} - (-1)^{|L|}\frac{\partial L}{\partial \pi v^i}\right)\mathrm{d}q^i + \left(\frac{\partial L}{\partial \pi\zeta^\alpha} - (-1)^{|L|}\frac{\partial L}{\partial \zeta^\alpha}\right)\mathrm{d}\theta^\alpha. \tag{5.19}$$

The Cartan graded 2-form is defined as the exact graded 2-form

$$\Omega_L = -\mathrm{d}\Theta_L \tag{5.20}$$

hence in local supercoordinates is written as

$$-\Omega_L = \left(\frac{\partial^2 L}{\partial q^i \partial v^j} - (-1)^{|L|}\frac{\partial^2 L}{\partial q^i \partial \pi v^j}\right)\mathrm{d}q^i \wedge \mathrm{d}q^j$$
$$+ \left(\frac{\partial^2 L}{\partial v^i \partial v^j} - (-1)^{|L|}\frac{\partial^2 L}{\partial v^i \partial \pi v^j}\right)\mathrm{d}v^i \wedge \mathrm{d}q^j$$

$$
\begin{aligned}
&+\left(\frac{\partial^2 L}{\partial \pi\zeta^\alpha \partial v^j} - (-1)^{|L|}\frac{\partial^2 L}{\partial \pi\zeta^\alpha \partial \pi v^j}\right) \mathrm{d}\pi\zeta^\alpha \wedge \mathrm{d}q^j \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \theta^\alpha \partial v^j} + \frac{\partial^2 L}{\partial \theta^\alpha \partial \pi v^j}\right) \mathrm{d}\theta^\alpha \wedge \mathrm{d}q^j \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \zeta^\alpha \partial v^j} + \frac{\partial^2 L}{\partial \zeta^\alpha \partial \pi v^j}\right) \mathrm{d}\zeta^\alpha \wedge \mathrm{d}q^j \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \pi v^i \partial v^j} + \frac{\partial^2 L}{\partial \pi v^i \partial \pi v^j}\right) \mathrm{d}\pi v^i \wedge \mathrm{d}q^j \\
&+\left(\frac{\partial^2 L}{\partial q^i \partial \zeta^\beta} - (-1)^{|L|}\frac{\partial^2 L}{\partial q^i \partial \zeta^\beta}\right) \mathrm{d}q^i \wedge \mathrm{d}\theta^\beta \\
&+\left(\frac{\partial^2 L}{\partial v^i \partial \zeta^\beta} - (-1)^{|L|}\frac{\partial^2 L}{\partial v^i \partial \zeta^\beta}\right) \mathrm{d}v^i \wedge \mathrm{d}\theta^\beta \\
&+\left(\frac{\partial^2 L}{\partial \pi\zeta^\alpha \partial \pi\zeta^\beta} - (-1)^{|L|}\frac{\partial^2 L}{\partial \pi\zeta^\alpha \partial \zeta^\beta}\right) \mathrm{d}\pi\zeta^\alpha \wedge \mathrm{d}\theta^\beta \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \theta^\alpha \partial \pi\zeta^\beta} + \frac{\partial^2 L}{\partial \theta^\alpha \partial \zeta^\beta}\right) \mathrm{d}\theta^\alpha \wedge \mathrm{d}\theta^\beta \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \zeta^\alpha \partial \pi\zeta^\beta} + \frac{\partial^2 L}{\partial \zeta^\alpha \partial \zeta^\beta}\right) \mathrm{d}\zeta^\alpha \wedge \mathrm{d}\theta^\beta \\
&-\left((-1)^{|L|}\frac{\partial^2 L}{\partial \pi v^i \partial \pi\zeta^\beta} + \frac{\partial^2 L}{\partial \pi v^i \partial \zeta^\beta}\right) \mathrm{d}\pi v^i \wedge \mathrm{d}\theta^\beta .
\end{aligned}
\tag{5.21}
$$

Therefore, matrix associated to Ω_L is of the form

$$
\Omega_L = \begin{pmatrix}
A_1 & A_2 & A_3 & B_1 & B_2 & B_3 \\
-A_2^t & 0 & 0 & B_4 & 0 & 0 \\
-A_3^t & 0 & 0 & B_5 & 0 & 0 \\
C_1 & C_4 & C_5 & D_1 & D_2 & D_3 \\
C_2 & 0 & 0 & D_2^t & 0 & 0 \\
C_3 & 0 & 0 & D_3^t & 0 & 0
\end{pmatrix}
\tag{5.22}
$$

where $C_i = -(-1)^{|L|}B_i^t$; in particular, Ω_L will be degenerate for every superfunction $L \in ST\mathcal{M}$.

5.3. *The super-Legendre transformation*

If Y is a vertical supervector field with respect to $\mathcal{T}$ (i.e. $Y \circ \tau^* = 0$) then $\Theta_L(Y) = 0$, and therefore Θ_L is a $\mathcal{T}$ semibasic graded 1-form, and since $\mathcal{T}$ is a submersion, it has associated a unique graded 1-form, $\hat{\Theta}_L$, along $\mathcal{T}$ [4]. In terms of the basis $\{\mathrm{d}\hat{q}^i, \mathrm{d}\hat{\theta}^\alpha\}$, $\hat{\Theta}_L$ has the same coordinates as Θ_L corresponding to the elements $\{\mathrm{d}q^i, \mathrm{d}\theta^\alpha\}$ (which is not a full basis of $\Omega^1\mathcal{A}(\mathcal{U})$), hence

$$
\hat{\Theta}_L = \left(\frac{\partial L}{\partial v^i} - (-1)^{|L|}\frac{\partial L}{\partial \pi v^i}\right)\mathrm{d}\hat{q}^i + \left(\frac{\partial L}{\partial \pi\zeta^\alpha} - (-1)^{|L|}\frac{\partial L}{\partial \zeta^\alpha}\right)\mathrm{d}\hat{\theta}^\alpha .
\tag{5.23}
$$

In analogy with non-graded geometry, see [7], the section $\mathcal{FL} : ST\mathcal{M} \to ST^*\mathcal{M}$ along $\mathcal{T}$ that corresponds to the graded 1-form $\hat{\Theta}_L$ could be considered as the Legendre transformation, but in view of the degeneracy of Ω_L for every $L \in ST\mathcal{M}$, we shall restrict our attention to the case when the super-Lagrangian $L \in T\mathcal{M} \subset ST\mathcal{M}$ (i.e. when L does not depend on the variables πv^i or $\pi\theta^\alpha$) and consider the restriction of $\mathcal{FL}$ to $T\mathcal{M}$.

Definition 5.4. If L is a super-Lagrangian in $T\mathcal{M}$, the super-Legendre transformation associated with L is the restriction of the map $\mathcal{FL}$ to $T\mathcal{M}$. We shall denote the super-Legendre transformation by FL. Hence

$$FL : T\mathcal{M} \to ST^*\mathcal{M}. \tag{5.24}$$

When $L \in T\mathcal{M}$ the matrix of Ω_L reduces to

$$\Omega_L = \begin{pmatrix} A_1 & A_2 & B_1 & B_2 \\ -A_2^t & 0 & B_4 & 0 \\ C_1 & C_4 & D_1 & D_2 \\ C_2 & 0 & D_2^t & 0 \end{pmatrix} \tag{5.25}$$

and to analyse its degeneracy it is necessary to consider the parity of L. If L is even then Ω_L is non-degenerate if, and only if, the matrices A_2 and D_2 are invertible; in other words, exactly when

$$\frac{\partial^2 L}{\partial v^i \partial v^j} \qquad \text{and} \qquad \frac{\partial^2 L}{\partial \zeta^\alpha \partial \zeta^\beta} \tag{5.26}$$

are invertible.

We also notice that if $|L| = 0$ then FL takes values in $T^*\mathcal{M}$. In fact, locally $FL = (fl, fl^*)$ is determined by the morphism of superalgebras $fl^* : T^*\mathcal{A}(\mathcal{U}) \to T\mathcal{A}(\mathcal{U})$ described by the relations:

$$\begin{aligned} q^i &\mapsto q^i \qquad & \theta^\alpha &\mapsto \theta^\alpha \\ p^i &\mapsto \frac{\partial L}{\partial v^i} \qquad & \eta^\alpha &\mapsto -\frac{\partial L}{\partial \zeta^\alpha} \end{aligned} \tag{5.27}$$

which, by the inverse function theorem [14], will be a local diffeomorphism when the Jacobian is invertible, and this happens exactly when (5.26) holds.

On the other hand, if L is odd, Ω_L is non-degenerate if, and only if, the off diagonal terms are non-degenerate. This implies that $m = n$ and that B_2 is invertible. In other words, that

$$\frac{\partial^2 L}{\partial \zeta^\alpha \partial v^j} \tag{5.28}$$

is invertible.

Unlike the even case, the super-Legendre transformation does not take values in $T^*\mathcal{M}$, but on the subsupermanifold of $ST^*\mathcal{M}$ of dimension $(m+n, n+m)$ obtained by imposing the conditions

$$p^i = 0 \qquad 1 \leqslant i \leqslant m \qquad \text{and} \qquad \eta^\alpha = 0 \qquad 1 \leqslant \alpha \leqslant n. \tag{5.29}$$

Moreover, locally FL is given by the assignments

$$\begin{aligned} q^i &\mapsto q^i \qquad & \theta^\alpha &\mapsto \theta^\alpha \\ \pi\eta^\alpha &\mapsto \frac{\partial L}{\partial \zeta^\alpha} \qquad & \pi p^i &\mapsto -\frac{\partial L}{\partial v^i} \end{aligned} \tag{5.30}$$

nevertheless, when $m = n$, again by the inverse function theorem, FL is a local diffeomorphism exactly when (5.28) holds. We have, therefore, proved the following.

Proposition 5.1. The super-Legendre transformation FL is a local diffeomorphism if, and only if, the graded form Ω_L is non-degenerate. In either case, we say that the super-Lagrangian L is regular.

The super-Legendre transformation has the same properties as the usual Legendre transformation [1].

Proposition 5.2. Let L be a super-Lagrangian in $ST\mathcal{A}$, then $\mathcal{FL}^*(\Theta_0) = \Theta_L$. Moreover, when $L \in T\mathcal{A}$ and one restricts Θ_L and Θ_0 to the appropriate subsupermanifolds (for instance to $T\mathcal{M}$ and $T^*\mathcal{M}$ respectively, when $|L| = 0$) then also $FL^*(\Theta_0) = \Theta_L$.

Proof. This is immediate from the local coordinate expressions. Let us simply remind the reader that, for instance, when $|L| = 0$ then $\mathcal{FL} = (fl, fl^*)$ is the morphism of supermanifolds associated to the morphism of superalgbras $fl^* : ST^*\mathcal{A}(\mathcal{U}) \to ST\mathcal{A}(\mathcal{U})$ given by

$$\begin{aligned} q^i &\mapsto q^i & \theta^\alpha &\mapsto \theta^\alpha \\ p^i &\mapsto \frac{\partial L}{\partial v^i} & \eta^\alpha &\mapsto -\frac{\partial L}{\partial \zeta^\alpha} \\ \pi\eta^\alpha &\mapsto \frac{\partial L}{\partial \pi\zeta^\alpha} & \pi p^i &\mapsto -\frac{\partial L}{\partial \pi v^i}. \end{aligned} \tag{5.31}$$

□

When L is a regular super-Lagrangian there exists a unique supervector field Γ_L in $\mathfrak{X}(\mathcal{M})$ such that

$$i_{\Gamma_L}\Omega_L = \mathrm{d}E_L \tag{5.32}$$

where the superenergy is defined by $E_L := \Delta L - L$ and Δ is the Liouville supervector field. Moreover, Γ_L is a super second order differential equation, see [12] for details.

Proposition 5.3. Let L be a super-Lagrangian in $T\mathcal{A}$ such that FL is a diffeomorphism (in such case we say L is hyperregular). Then $V = (FL^{-1})^* \circ \Gamma_L \circ FL^*$ is a Hamiltonian supervector field with Hamiltonian $H := (FL^{-1})^* E_L$. Reciprocally, if H is the super function $H := (FL^{-1})^* E_L$, then the Hamiltonian supervector field V associated to L is FL-related to Γ_L.

Proof. Let X be a supervector field on $T^*\mathcal{M}$. Since FL is a diffeomorphism there exists $Y \in \mathfrak{X}(T\mathcal{A})$ such that $X = (FL^{-1})^* \circ Y \circ FL^*$. Using lemma 3.1 twice we have

$$\begin{aligned} i_V\Omega_0(X) &= \Omega_0(V, X) = (FL^{-1})^*[FL^*(\Omega_0)(\Gamma_L, Y)] = (FL^{-1})^*[i_{\Gamma_L}\Omega_L(Y)] \\ &= [\mathrm{d}(FL^{-1})^*(E_L)](X) = \mathrm{d}H(X) \end{aligned} \tag{5.33}$$

and the first assertion follows.

As for the second statement, we consider $Z = (FL^{-1})^* \circ \Gamma_L \circ FL^*$; then the previous argument gives that $i_Z\Omega_0 = \mathrm{d}H$, and since Ω_0 is non-degenerate, $Z = V$, and the proposition is proved.

Moreover, since Γ_L is a super SODE then $i_{\Gamma_L}\Theta_L = \Delta(L) =: A$, and the same argument gives us $\Theta_0(V) = (FL^{-1})^* A$ when L is hyperregular. The correspondence between the Lagrangian and Hamiltonian formulations in supermechanics is clear.

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